Basic Theorem of Continuity

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September 21, 2017

Abstract

In this article I present the basic theorem of continuity, with some examples, for real valued functions of one variable. The theorem says that the concept of continuity "behaves well" with respect to algebraic operations between functions. The article can be used as a chapter in an introductory calculus textbook.

If we look at graphs of functions we know what continuity means: we can draw the whole graph of the function, from the beginning to the end, without ever lifting the pencil from the paper. But, how to formalize this in a mathematical theory? We need the concept of approximation. The level of approximation of some real number c is given with a number $\epsilon > 0$. All numbers that approximate c up to ϵ satisfy the following inequality:

$$|x - c| < \epsilon$$

With the help of this concept we define a continuous function: a function f defined on an interval I is continuous at a point $c \in I$, if for every $\epsilon > 0$ there exists at least one $\delta > 0$ such that:

$$|x - c| < \delta \implies |f(x) - f(c)| < \epsilon$$

In other words: for every ϵ-approximation (no matter how correct) of $f(c)$ we can always find a $\delta > 0$ such that the implication holds.

For example, let's look at the function $f(x) = x$ with $\epsilon > 0$:

$$|f(x) - f(c)| = |x - c| < \epsilon$$

If we take $\delta = \epsilon$ and look at the implication from the continuity definition, we conclude that $f(x) = x$ is continuous.

For another example, let's look at the function:

$$f(x) = \frac{1}{x}$$

and assume that f has a discontinuity at some point c. That would mean that there exists $\epsilon > 0$ such that for every $\delta > 0$ the following holds:

$$|x - c| < \delta \wedge |f(x) - f(c)| > \epsilon$$

but:

$$|f(x) - f(c)| = \left|\frac{1}{x} - \frac{1}{c}\right| = \frac{|x-c|}{|xc|} > \epsilon$$

therefore:

$$|x - c| > \epsilon|xc|$$

This holds true for every $\delta > 0$ even for $\delta = \epsilon|xc|$:

$$\epsilon|xc| > |x - c| > \epsilon|xc|$$

Which implies:

$$|xc| > |xc|$$

which is a contradiction, and therefore our assumption was incorrect, $f(x) = \frac{1}{x}$ is a continuous function.

What about the function defined as:

$$f(x) = \frac{x^2 + 1}{x}, x \neq 0$$

We can write the formula for f as:

$$f(x) = x + \frac{1}{x}$$

and we see that the function f is the result of addition of two continuous functions. Let's ask more generally: if f, g are continuous, is $h = f + g$ continuous? The continuity of f and g means that for every $\epsilon > 0$ there exist numbers $\delta_f, \delta_g > 0$ such that:

$$|x - c| < \delta_f \implies |f(x) - f(c)| < \epsilon$$

and:

$$|x - c| < \delta_g \implies |g(x) - g(c)| < \epsilon$$

If we take $\delta = \min\{\delta_f, \delta_g\}$:

$$|x - c| < \delta \implies |f(x) - f(c)| < \epsilon \wedge |g(x) - g(c)| < \epsilon$$

But:

$$|h(x) - h(c)| = |f(x) - f(c) - g(x) + g(c)| \leq |f(x) - f(c)| + |g(x) - g(c)| < 2\epsilon$$

Because $\epsilon > 0$ is arbitrarily small we conclude that $h = f + g$ is a continuous function if f and g are continuous. For another statement, which is proved in a similar way, if f, g are continuous, so is $h = f \circ g$, the composition of the two functions. The proof of this statement is left as an exercise to the reader. Let's now look at the function:

$$h(x) = \frac{f(x)}{g(x)}$$

If f, g are continuous, so is h. To see this we can write $h(x)$ as:

$$h(x) = f(x) \cdot \frac{1}{g(x)}$$

We know that $w(x) = \frac{1}{x}$ is continuous, which means that $w \circ g = \frac{1}{g}$ is continuous, which we know from the previous exercise. Therefore, if we prove that $h = fg$ is continuous whenever f, g are continuous, we will prove that $h(x) = f/g$ is continuous whenever f, g are continuous. To prove this let's write:

$$h(x) - h(c) = f(x)g(x) - f(c)g(c) = f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c) = f(x)[g(x) - g(c)] + g(c)[f(x) - f(c)]$$

We know that:

$$|f(x) - f(c)| < \epsilon$$

and:

$$|g(x) - g(c)| < \epsilon$$

But what about factors like $f(x), g(c)$. We can imagine the situation in which $|f(x)||g(x) - g(c)|$ is large when $|g(x) - g(c)|$ is arbitrarily small. To solve this we need another concept, that of a bounded function. We say that a function f is bounded if for every point in the surroundings of one point c there exist a number $M > 0$ such that $|f(x)| < M$. Let's suppose that f is continuous at a point c. Then:

$$|f(x)| = |f(x) - f(c) + f(c)| \leq |f(x) - f(c)| + |f(c)| < \epsilon + |f(c)|$$

If we write $M = \epsilon + |f(c)|$ we see that $|f(x)| < M$ and therefore every continuous function is a bounded function. If we use this result on our function $h = fg$ we see that:

$$|h(x) - h(c)| < M\epsilon + N\epsilon$$

Because $\epsilon > 0$ is arbitrarily small and M, N are constants, we conclude that $h = fg$ is continuous whenever f, g are continuous. For an example we now know that functions like:

$$f(x) = \frac{1 + x^2}{1 + x}$$

are continuous.

To sum up our explorations we write the basic theorem of continuity which says that the concept of continuity "behaves well" with respect to algebraic operations between functions.

Theorem: Let f, g be functions defined on an interval I such that f, g are continuous on the same interval. Then:

1. $f + g$ is continuous on I
2. $f - g$ is continuous on I
3. λf is continuous on I for every constant number λ
4. fg is continuous on I
5. $f(x)/g(x)$ is continuous on I if $g(x) \neq 0$ for every $x \in I$